Modal Logic of Forcing Classes

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Outline

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Modal Logic Background

Modal Axioms

$$
\begin{array}{ll}
\mathrm{K} & \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi) \\
\mathrm{T} & \Box\varphi \to \varphi \\
4 & \Box\varphi \to \Box\Box\varphi \\
.2 & \Diamond\Box\varphi \to \Box\Diamond\varphi \\
.3 & (\Diamond\varphi \wedge \Diamond\psi) \to \Diamond[(\varphi \wedge \Diamond\psi) \vee (\psi \wedge \Diamond\varphi)] \\
5 & \Diamond\Box\varphi \to \varphi
\end{array}
$$

Modal Theories

$$
\begin{array}{rcl}
\mathsf{S4} & = & \mathrm{K} + \mathrm{T} + 4 \\
\mathsf{S4.2} & = & \mathrm{K} + \mathrm{T} + 4 + .2 \\
\mathsf{S4.3} & = & \mathrm{K} + \mathrm{T} + 4 + .3 \\
\mathsf{S5} & = & \mathrm{K} + \mathrm{T} + 4 + 5
\end{array}
$$

Modal Logic Background

Soundness and Completeness with respect to Frame Classes

If F is a frame, a modal assertion is *valid for F* if it is true at all worlds of all Kripke models having frame F, and it is *valid for F at w* if it is true at w in all Kripke models having frame F.

If $\mathcal{C}$ is a class of frames, a modal theory is *sound with respect to $\mathcal{C}$* if every assertion in the theory is valid for every frame in $\mathcal{C}$.

A modal theory is *complete with respect to $\mathcal{C}$* if every assertion valid for every frame in $\mathcal{C}$ is in the theory.

Finally, a modal theory is *characterized by $\mathcal{C}$* (equivalently, $\mathcal{C}$ *characterizes* the modal theory) if it is both sound and complete with respect to $\mathcal{C}$[11, p. 40].

Modal Logic Background

Theorem

The modal logic S4.3 is characterized by the class of finite linear pre-order frames. That is, a modal assertion is derivable in S4.3 if and only if it holds in all Kripke models having a finite linear pre-ordered frame. Cf. [2]

Theorem

[[9, theorem 11]] The modal logic S4.2 is characterized by the class of finite pre-Boolean algebras. That is, a modal assertion is derivable in S4.2 if and only if it holds in all Kripke models having a finite pre-Boolean algebra frame.

Theorem

The modal logic S5 is characterized by the class of finite equivalence relations with one equivalence class (a single cluster).

Valid Modal Logic Principles for Forcing Classes

Definitions

A set-theoretic sentence ψ is Γ-*forceable* or Γ-*possible*, written $\Diamond_\Gamma \psi$ (or simply $\Diamond \psi$), if ψ holds in a forcing extension by some forcing notion in Γ, and ψ is Γ-*necessary*, written $\Box_\Gamma \psi$ (or simply $\Box \psi$), if ψ holds in all forcing extensions by forcing notions in Γ.

For any forcing class Γ, every assignment $p_i \mapsto \psi_i$ of the propositional variables p_i to set-theoretical assertions ψ_i extends recursively to a Γ *forcing translation* $\mathrm{H}:\mathcal{L}_\Box \to \mathcal{L}_\in$. $H(\varphi)$ is called a *substitution instance* of the modal assertion φ. In this terminology, the *modal logic of* Γ *forcing* over a model of set theory W is the set

$$\{\,\varphi \in \mathcal{L}_\Box \mid W \models H(\varphi) \text{ for all } \Gamma \text{ forcing translations } H\,\}.$$

A formula in this set is said to be a *valid principle* of Γ forcing.

Valid Modal Logic Principles for Forcing Classes

Definitions

A forcing class Γ is *reflexive* if in every model of set theory, Γ contains the trivial forcing poset.

The class Γ is *transitive* if it closed under finite iterations, in the sense that if $\mathbb{Q} \in \Gamma$ and $\dot{\mathbb{R}} \in \Gamma^{V^{\mathbb{Q}}}$, then $\mathbb{Q} * \dot{\mathbb{R}} \in \Gamma$.

The class Γ is *closed under product forcing* if, necessarily, whenever $\mathbb{Q}$ and $\mathbb{R}$ are in Γ, then so is $\mathbb{Q} \times \mathbb{R}$. Related to this, Γ is *persistent* if, necessarily, members of Γ are Γ necessarily in Γ; that is, if $\mathbb{P}, \mathbb{Q} \in \Gamma$ implies $\mathbb{P} \in \Gamma^{V^{\mathbb{Q}}}$ in all models.

The class Γ is *directed* if whenever $\mathbb{P}, \mathbb{Q} \in \Gamma$, then there is $\mathbb{R} \in \Gamma$, such that both $\mathbb{P}$ and $\mathbb{Q}$ are factors of $\mathbb{R}$ by further Γ forcing, that is, if $\mathbb{R}$ is forcing equivalent to $\mathbb{P} * \dot{\mathbb{S}}$ for some $\dot{\mathbb{S}} \in \Gamma^{V^{\mathbb{P}}}$ and also equivalent to $\mathbb{Q} * \dot{\mathbb{T}}$ for some $\dot{\mathbb{T}} \in \Gamma^{V^{\mathbb{Q}}}$.

The class Γ has the *linearity property* if for any two forcing notions $\mathbb{P}, \mathbb{Q}$, then one of them is forcing equivalent to the other one followed by additional Γ forcing; that is, either $\mathbb{P}$ is forcing equivalent to $\mathbb{Q} * \dot{\mathbb{R}}$ for some $\dot{\mathbb{R}} \in \Gamma^{V^{\mathbb{Q}}}$ or $\mathbb{Q}$ is forcing equivalent to $\mathbb{P} * \dot{\mathbb{R}}$ for some $\dot{\mathbb{R}} \in \Gamma^{V^{\mathbb{Q}}}$. Combining these notions, we define that Γ is a *linear forcing class* if Γ is reflexive, transitive and has the linearity property.

Valid Modal Logic Principles for Forcing Classes

Theorem

1. S4 *is valid for any reflexive transitive forcing class.*
2. S4.2 *is valid for any reflexive transitive directed forcing class.*
3. S4.3 *is valid for any linear forcing class.*

Γ — Labeling

Definition

Suppose that F is a transitive reflexive frame with initial world w_0. A Γ-labeling of this rooted frame for a model of set theory W is an assignment to each node w in F an assertion Φ_w in the language of set theory, such that

1. *The statements Φ_w form a mutually exclusive partition of truth in the Γ forcing extensions of W, meaning that every such extension $W[G]$ satisfies exactly one Φ_w.*
2. *Any Γ forcing extension $W[G]$ in which Φ_w is true satisfies $\Diamond \Phi_u$ if and only if $w \leq_F u$.*
3. *$W \models \Phi_{w_0}$, where w_0 is the given initial world of F.*

Γ – *Labeling*

Lemma

Suppose that $w \mapsto \Phi_w$ is a Γ-labeling for a model of set theory W of a finite transitive reflexive frame F with initial world w_0. Then for any Kripke model M having frame F, there is an assignment of the propositional variables to set-theoretic assertions $p \mapsto \psi_p$ such that for any modal assertion $\varphi(p_0, \ldots, p_k)$,

$$(M, w_0) \models \varphi(p_0, \ldots, p_k) \quad \textit{iff} \quad W \models \varphi(\psi_{p_0}, \ldots, \psi_{p_k}).$$

In particular, any modal assertion φ that fails at w_0 in M also fails in W under the Γ forcing interpretation. Consequently, the modal logic of Γ forcing over W is contained in the modal logic of assertions valid in F at w_0.

Proof.

Suppose that $w \mapsto \Phi_w$ is a Γ-labeling of F for W, and suppose that M is a Kripke model with frame F. Thus, we may view each $w \in F$ as a propositional world in M. For each propositional variable p, let $\psi_p = \bigvee\{\, \Phi_w \mid (M, w) \models p \,\}$. We prove, a fortiori, that whenever $W[G]$ is a Γ forcing extension of W and $W[G] \models \Phi_w$, then

$$(M, w) \models \varphi(p_0, \ldots, p_k) \quad \text{iff} \quad W[G] \models \varphi(\psi_{p_0}, \ldots, \psi_{p_k}).$$

The proof is by induction on the complexity of φ. □

Control Statements

Switches and Buttons

Suppose that Γ is a reflexive transitive forcing class.
A *switch* for Γ is a statement s such that both s and $\neg s$ are Γ necessarily possible.
A *button* for Γ is a statement b that is Γ necessarily possibly necessary.
In the case that S4.2 is valid for Γ, this is equivalent to saying that b is possibly necessary. The button b is *pushed* when $\Box b$ holds, and otherwise it is *unpushed*.
A finite collection of buttons and switches (or other controls of this type) is *independent* if, necessarily, each can be operated without affecting the truth of the others.

Control Statements

Ratchets

A sequence of first-order statements r_1, $r_2, \ldots r_n$ is a *ratchet* for Γ of length n if each is an unpushed pure button for Γ, each necessarily implies the previous, and each can be pushed without pushing the next. This is expressed formally as follows:

$$\begin{array}{l} \neg r_i \\ \Box(r_i \to \Box\, r_i) \\ \Box(r_{i+1} \to r_i) \\ \Box[\neg r_{i+1} \to \Diamond(r_i \wedge \neg r_{i+1})] \end{array}$$

A ratchet is **unidirectional:** any further Γ forcing can only increase the ratchet value or leave it the same.

A ratchet is *uniform* if there is a formula $r(x)$ with one free variable, such that $r_\alpha = r(\alpha)$. Every finite length ratchet is uniform. A ratchet is *continuous*, if for every limit ordinal $\lambda < \delta$, the statement r_λ is equivalent to $\forall\alpha{<}\lambda\, r_\alpha$.

A *long ratchet* is a uniform ratchet $\langle r_\alpha \mid 0 < \alpha < \mathrm{ORD}\rangle$ of length ORD, with the additional property that no Γ forcing extension satisfies all r_α, so that every Γ extension exhibits some ordinal ratchet value.

Sample Result

Theorem

If Γ is a reflexive transitive forcing class having arbitrarily long finite ratchets over a model of set theory W, mutually independent with arbitrarily large finite families of switches, then the valid principles of Γ forcing over W are contained within the modal theory S4.3.

Proof.

Suppose that Γ is a reflexive transitive forcing class with arbitrarily long finite ratchets, mutually independent of switches over a model of set theory W. By the theorem on valid principles of forcing classes, any modal assertion not in S4.3 must fail at an initial world of a Kripke model M built on a finite pre-linear order frame, consisting of a finite increasing sequence of n clusters of mutually accessible worlds $w_0^k, w_1^k, \ldots, w_{n_k-1}^k$. The frame order is simply $w_i^k \leq w_j^s$ if and only if $k \leq s$. We may assume that all clusters have the same size $n_k = 2^m$ for fixed m.
Let $r_1, \ldots, r_n$ be a ratchet of length n for Γ over W, mutually independent from the m many switches $s_0, \ldots, s_{m-1}$. We may assume that all switches are off in W. Let $\bar{r}_k$ be the assertion that the ratchet value is exactly k, so that $\bar{r}_0 = \neg r_1$, $\bar{r}_k = r_k \wedge \neg r_{k+1}$ for $1 \leq k < n$ and $\bar{r}_n = r_n$, and let $\bar{s}_j$ assert for $j < 2^m$ that the pattern of switches accords with the m binary digits of j. The required Γ labeling assigns to each world w_j^k, where $k < n$ and $j < 2^m$, the assertion that the ratchet value is exactly k and the switches exhibit pattern j. □

Sample Result

Theorem

If Γ is a reflexive transitive forcing class having a long ratchet over a model of set theory W, then the valid principles of Γ forcing over W are contained within the modal theory S4.3.

Proof.

Suppose that $\langle r_\alpha \mid 0 < \alpha < \mathrm{ORD}\rangle$ is a long ratchet over W, that is, a uniform ratchet control of length ORD, such that no Γ extension satisfies every r_α. We may assume the ratchet is continuous. It suffices by theorem 7 to produce arbitrarily long finite ratchets independent from arbitrarily large finite families of switches. To do this, we shall divide the ordinals into blocks of length ω, and think of the position within one such a block as determining a switch pattern and the choice of block itself as another ratchet. Specifically, every ordinal can be uniquely expressed in the form $\omega \cdot \alpha + k$, where $k < \omega$, and we think of this ordinal as being the kth element in the αth block. Let s_i be the statement that if the current ratchet value is exactly $\omega \cdot \alpha + k$, then the ith binary bit of k is 1. Let v_α be the assertion $r_{\omega\cdot\alpha}$, which expresses that the current ratchet value is in the αth block of ordinals of length ω or higher. Since we may freely increase the ratchet value to any higher value, we may increase the value of k while staying in the same block of ordinals, and so the v_α form themselves a ratchet, mutually independent of the switches s_i. Thus, by the previous theorem, the valid principles of Γ forcing over W are contained within S4.3. $\square$

Sample Result

Theorem

If ZFC *is consistent, then the* ZFC*-provably valid principles of collapse forcing* $\mathrm{Coll} = \{\mathrm{Coll}(\omega, \theta) | \theta \in \mathrm{ORD}\}$ *are exactly those in* S4.3.

Proof.

(Sketch) For the lower bound, Coll is easily seen to be a linear forcing class (it includes trivial forcing, hence reflexive; also, $\mathrm{Coll}(\omega, <\theta) * \mathrm{Coll}(\omega, <\lambda)$ is forcing equivalent to $\mathrm{Coll}(\omega, < \max\{\theta, \lambda\})$, so Coll is transitive.
For the upper bound, we shall show that Coll admits a long ratchet over the constructible universe L. For each non-zero ordinal α, let r_α be the statement "$\aleph_\alpha^{\mathrm{L}}$ is countable." These statements form a long ratchet for collapse forcing over the constructible universe L, since any collapse extension $\mathrm{L}[G]$ collapses an initial segment of the cardinals of L to ω, and in any such extension in which $\aleph_\alpha^{\mathrm{L}}$ is not yet collapsed, the forcing to collapse it will not yet collapse $\aleph_{\alpha+1}^{\mathrm{L}}$. Thus, by the previous theorem, the valid principles of collapse forcing over L are contained within S4.3. So the valid principles of collapse forcing over L are precisely S4.3, and if ZFC is consistent, then the ZFC-provably valid principles of collapse forcing are exactly S4.3. $\square$

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